An improved Hardy-Trudinger-Moser inequality

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Abstract

Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$, $\mathscr{H}$ be the completion of $C_0^\infty(\mathbb{B})$ under the norm

$$\|u\|_{\mathscr{H}} = \left(\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx\right)^{1/2}, \quad \forall u \in C_0^\infty(\mathbb{B}).$$

Denote $\lambda_1(\mathbb{B}) = \inf_{u\in\mathscr{H},\,\|u\|_2=1} \|u\|_{\mathscr{H}}^2$, where $\|\cdot\|_2$ stands for the $L^2(\mathbb{B})$-norm. Using blow-up analysis, we prove that for any α, $0 \le \alpha < \lambda_1(\mathbb{B})$,

$$\sup_{u\in\mathscr{H},\,\|u\|_{\mathscr{H}}^2-\alpha\|u\|_2^2\le 1} \int_{\mathbb{B}} e^{4\pi u^2} dx < +\infty,$$

and that the above supremum can be attained by some function $u \in \mathscr{H}$ with $\|u\|_{\mathscr{H}}^2 - \alpha\|u\|_2^2 = 1$. This improves an earlier result of G. Wang and D. Ye [29].

Key words: Hardy-Trudinger-Moser inequality, Trudinger-Moser inequality, blow-up analysis
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1. Introduction

Let $\mathbb{B}$ be the unit disc in $\mathbb{R}^2$ and $W_0^{1,2}(\mathbb{B})$ be the usual Sobolev space. The Trudinger-Moser inequality [34, 24, 23, 28, 22] says that for any $\beta \le 4\pi$,

$$\sup_{u\in W_0^{1,2}(\mathbb{B}),\,\|\nabla u\|_2\le 1} \int_{\mathbb{B}} e^{\beta u^2} dx < \infty. \tag{1}$$

Here and throughout this paper we denote the $L^p(\mathbb{B})$-norm by $\|\cdot\|_p$. This inequality is sharp in the sense that for any $\beta > 4\pi$, the integrals in (1) are still finite but the supremum is infinity. It is a very powerful tool in the problem of prescribed Gaussian curvature and partial differential equations.

Another important inequality in analysis is the Hardy inequality, namely

$$\int_{\mathbb{B}} |\nabla u|^2 dx \ge \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx, \quad \forall u \in W_0^{1,2}(\mathbb{B}).$$

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The Hardy inequality was improved in many ways. It was proved by H. Brezis and M. Marcus [7] that there exists some constant C such that

$$\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx \geq C \int_{\mathbb{B}} u^2 dx, \quad \forall u \in W_0^{1,2}(\mathbb{B}). \tag{2}$$

Further improvements known as the Hardy-Sobolev inequalities were done by Maz'ya ([21], Corollary 3, Section 2.1.6), Mancini-Sandeep [19], Adimurthi-do Ó-Tintarev [2], and Mancini-Sandeep-Tintarev [20]. In view of (2),

$$\|u\|_{\mathscr{H}} = \left(\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx \right)^{1/2}$$

defines a norm on $C_0^\infty(\mathbb{B})$. Let $\mathscr{H}$ be the completion of $C_0^\infty(\mathbb{B})$ under the norm $\|\cdot\|_{\mathscr{H}}$. Clearly $\mathscr{H}$ is a Hilbert space. By a result of Mancini-Sandeep ([19], the inequality (1.2)), we can see that for any $p > 1$, there exists a constant $C_p > 0$ such that

$$\|u\|_p \leq C_p \|u\|_{\mathscr{H}}, \quad \forall u \in \mathscr{H}. \tag{3}$$

This is also obtained by Wang-Ye [29]. Thus we have

$$W_0^{1,2}(\mathbb{B}) \subset \mathscr{H} \subset \cap_{p\geq 1} L^p(\mathbb{B}).$$

Obviously $\mathscr{H} \not\subset L^\infty(\mathbb{B})$. In view of (1), one can expect a Hardy-Trudinger-Moser inequality, namely

$$\sup_{u\in\mathscr{H},\, \|u\|_{\mathscr{H}}\leq 1} \int_{\mathbb{B}} e^{4\pi u^2} dx < +\infty. \tag{4}$$

This was done by Wang-Ye by using blow-up analysis in [29], where the existence of extremal function for (4) was also obtained. The inequality (4) was further extended by C. Tintarev [27] to a generalized Euclidean version by using Ground state transform, and by Mancini-Sandeep-Tintarev [20] to a hyperbolic space version via a rearrangement argument. Compared with (4), another kind of singular Trudinger-Moser inequalities were obtained by Adimurthi-Sandeep [3], Adimurthi-Yang [4], and de Souza-do Ó [12].

Motivated by the works of Adimurthi-Druet [1], Y. Yang [30, 31, 32, 33] and C. Tintarev [27], we aim to rewrite (4) with $\|u\|_{\mathscr{H}}$ replaced by certain equivalent norm on $\mathscr{H}$. To clarify this problem, we define

$$\lambda_1(\mathbb{B}) = \inf_{u\in\mathscr{H},\, u\not\equiv 0} \frac{\|u\|_{\mathscr{H}}^2}{\|u\|_2^2}. \tag{5}$$

By (3) and a variational direct method, we have that $\lambda_1(\mathbb{B})$ can be attained by some function $u \in \mathscr{H}$ with $\|u\|_2 = 1$. In particular, $\lambda_1(\mathbb{B}) > 0$. In fact, $\lambda_1(\mathbb{B})$ is the first eigenvalue of the Hardy-Laplace operator, namely

$$\mathscr{L}_H = -\Delta - \frac{1}{(1-|x|^2)^2}.$$

For any α, $0 \leq \alpha < \lambda_1(\mathbb{B})$ and any $u \in \mathscr{H}$, we denote

$$\|u\|_{1,\alpha} = \left(\|u\|_{\mathscr{H}}^2 - \alpha \|u\|_2^2 \right)^{1/2}. \tag{6}$$

Clearly $\|\cdot\|_{1,\alpha}$ is equivalent to $\|\cdot\|_{\mathscr{H}}$. Our main result is the following:

Theorem 1. *Let $\mathbb{B}$ be the unit ball in $\mathbb{R}^2$, $\lambda_1(\mathbb{B})$ be defined as in (5). Then for any $\beta \le 4\pi$ and any α, $0 \le \alpha < \lambda_1(\mathbb{B})$, the supremum*

$$\sup_{u\in\mathscr{H},\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{\beta u^2} dx$$

can be attained by some function $u_0 \in \mathscr{H}$ with $\|u_0\|_{1,\alpha} = 1$, where $\|\cdot\|_{1,\alpha}$ is defined as in (6).

An interesting consequence of Theorem 1 is the following weak form of the Hardy-Trudinger-Moser inequality.

Corollary 2. *Let $\lambda_1(\mathbb{B})$ be defined as in (5). Then for any α, $0 \le \alpha < \lambda_1(\mathbb{B})$, there exists a constant $C > 0$ depending only on α such that*

$$\int_{\mathbb{B}} |\nabla u|^2 dx - \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2} dx - \alpha \int_{\mathbb{B}} u^2 dx - 16\pi \log \int_{\mathbb{B}} e^u dx \ge -C, \quad \forall u \in W_0^{1,2}(\mathbb{B}).$$

Following the lines of Y. Li [15], Adimurthi-Druet [1], Yang [31], and Wang-Ye [29], we prove Theorem 1 by using blow-up analysis. We remark that Wang-Ye [29] solved (4) by using a result of Carleson-Chang [10] in addition to standard blow-up analysis. This method was originally used by Li-Liu-Yang [16]. In this paper, we shall employ the capacity estimate introduced by Y. Li [15], instead of Carleson-Chang's result. It would be interesting to extend our Theorem 1 to the case involving $L^p(\mathbb{B})$-norm as in [18].

Earlier works in this direction were due to Carleson-Chang [10], M. Struwe [25], F. Flucher [13], K. Lin [17], Ding-Jost-Li-Wang [11], and Adimurthi-Struwe [5]. The remaining part of this paper is organized as follows. In Section 2, we give several preliminary lemmas; In Section 3, we prove Theorem 1.

2. Preliminary results

In this section, we list several properties of the space $\mathscr{H}$. Let

$$\mathscr{S}_0 = \left\{ u \in C_0^\infty(B) : u(x) = u(r), u'(r) \le 0, \text{ where } r = |x| \right\}$$

and $\mathscr{S}$ be the completion of $\mathscr{S}_0$ under the norm $\|\cdot\|_{\mathscr{H}}$. The following embedding theorem was proved by Wang-Ye [29]:

Lemma 3. *$\mathscr{S}$ is embedded continuously in $W^{1,2}_{\mathrm{loc}}(\mathbb{B}) \cap C^{0,\frac{1}{2}}_{\mathrm{loc}}(\mathbb{B} \setminus \{0\})$. Moreover, $\mathscr{S}$ is embedded compactly in $L^p(\mathbb{B})$ for any $p \ge 1$.*

The second important property of $\mathscr{H}$ is an embedding of Orlicz type, namely

Lemma 4. *For any $p > 1$ and any $u \in \mathscr{H}$, there holds*

$$\int_{\mathbb{B}} e^{pu^2} dx < +\infty.$$

Proof. Fix $p > 1$ and $u \in \mathscr{H}$. Since $C_0^\infty(\mathbb{B})$ is dense in $\mathscr{H}$, we take $u_0 \in C_0^\infty(\mathbb{B})$ such that $\|u - u_0\|^2_{\mathscr{H}} < \pi/p$. Using an inequality $2ab \le a^2 + b^2$ twice, we have

$$\int_{\mathbb{B}} e^{pu^2} dx \le \int_{\mathbb{B}} e^{4p(u-u_0)^2} dx + \int_{\mathbb{B}} e^{4pu_0^2} dx.$$

By ([29], Theorem 1), we have

$$\int_{\mathbb{B}} e^{4p(u-u_0)^2}dx < +\infty.$$

Since u_0 is uniformly bounded in $\mathbb{B}$, we have $\int_{\mathbb{B}} e^{4pu_0^2}dx < +\infty$. This gives the desired estimate. $\square$

Finally we state an obvious but very important property of $\mathscr{H}$.

Lemma 5. *Suppose $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{B})$, $v \in \mathscr{H}$ and v is radially symmetric. If there exists some r, $0 < r < 1$, such that $u = v$ on $\mathbb{B} \setminus \mathbb{B}_r$, then $u \in \mathscr{H}$.*

Proof. It follows from Lemma 3 that $v \in W^{1,2}_{\mathrm{loc}}(\mathbb{B})$. Clearly we have $u - v \in W^{1,2}_0(\mathbb{B}_r) \subset W^{1,2}_0(\mathbb{B})$. This leads to $u = (u - v) + v \in \mathscr{H}$. $\square$

3. Proof of Theorem 1

In this section, we prove Theorem 1 by using a blow-up scheme similar to that of Wang-Ye [29], and thereby follow closely Y. Li [15], Adimurthi-Druet [1], and Yang [31]. The proof will be divided into several subsections.

3.1. The subcritical case

In this subsection, we prove that the subcritical Trudinger-Moser functional $J(u) = \int_{\mathbb{B}} e^{\gamma u^2}dx$ has a maximizer in the function space $\{u \in \mathscr{H} : \|u\|_{1,\alpha} \le 1\}$ for any $\gamma < 4\pi$ and any α, $0 \le \alpha < \lambda_1(\mathbb{B})$.

Proposition 6. *Let $0 \le \alpha < \lambda_1(\mathbb{B})$. For any $\epsilon > 0$, there exists $u_\epsilon \in \mathscr{S} \cap C^\infty(\mathbb{B}) \cap C^0(\overline{\mathbb{B}})$ such that $\|u_\epsilon\|_{1,\alpha} = 1$ and*

$$\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2}dx = \sup_{u\in\mathscr{H},\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2}dx. \tag{7}$$

Moreover, u_ϵ satisfies the following Euler-Lagrange equation

$$\begin{cases} -\Delta u_\epsilon - \frac{u_\epsilon}{(1-|x|^2)^2} - \alpha u_\epsilon = \frac{1}{\lambda_\epsilon} u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} \text{ in } \mathbb{B}, \\ \lambda_\epsilon = \int_{\mathbb{B}} u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2}dx. \end{cases} \tag{8}$$

Furthermore, we have

$$\liminf_{\epsilon\to 0} \lambda_\epsilon > 0. \tag{9}$$

Proof. We first claim that for any $\gamma < 4\pi$, there holds

$$\sup_{u\in\mathscr{H},\, \|u\|_{\mathscr{H}}\le 1} \int_{\mathbb{B}} e^{\gamma u^2}dx < +\infty. \tag{10}$$

To see this, we use an argument of radially decreasing rearrangement with respect to the standard hyperbolic metric $dv = \frac{1}{(1-|x|^2)^2}dx$. For any $u \in C_0^\infty(\mathbb{B})$, we let u^* be the radially decreasing rearrangement of $|u|$ with respect to the standard hyperbolic metric. It follows from [6] that

$$\int_{\mathbb{B}} |\nabla u^*|^2 dx \le \int_{\mathbb{B}} |\nabla u|^2 dx, \quad \int_{\mathbb{B}} \frac{u^{*2}}{(1-|x|^2)^2}dx = \int_{\mathbb{B}} \frac{u^2}{(1-|x|^2)^2}dx.$$

Clearly we have $\|u^*\|_{\mathscr{H}} \le \|u\|_{\mathscr{H}}$. This leads to

$$\sup_{u\in C_0^\infty(\mathbb{B}),\, \|u\|_{\mathscr{H}}\le 1} \int_{\mathbb{B}} e^{\gamma u^2} dx \le \sup_{u\in \mathscr{S},\, \|u\|_{\mathscr{H}}\le 1} \int_{\mathbb{B}} e^{\gamma u^2} dx.$$

In view of Lemma 3, for any $u \in \mathscr{H}$ with $\|u\|_{\mathscr{H}} \le 1$, there exists a sequence of functions $u_j \in C_0^\infty(\mathbb{B})$ such that $\|u_j\|_{\mathscr{H}} \le 1$, $u_j \to u$ in $\mathscr{H}$ and $u_j \to u$ a. e. in $\mathbb{B}$. Thus

$$\int_{\mathbb{B}} e^{\gamma u^2} dx \le \limsup_{j\to\infty} \int_{\mathbb{B}} e^{\gamma u_j^2} dx.$$

Hence

$$\sup_{u\in \mathscr{H},\, \|u\|_{\mathscr{H}}\le 1} \int_{\mathbb{B}} e^{\gamma u^2} dx = \sup_{u\in \mathscr{S},\, \|u\|_{\mathscr{H}}\le 1} \int_{\mathbb{B}} e^{\gamma u^2} dx,$$

which together with ([29], Theorem 3) implies (10).

Now let $0 < \epsilon < 4\pi$ be fixed. Note that

$$\begin{aligned}
\int_{\mathbb{B}} u^{*2} dx &= \int_{\mathbb{B}} u^{*2}(1-|x|^2)^2 dv \\
&\ge \int_{\mathbb{B}} (u^2(1-|x|^2)^2)^* dv \\
&= \int_{\mathbb{B}} u^2(1-|x|^2)^2 dv \\
&= \int_{\mathbb{B}} u^2 dx,
\end{aligned}$$

and that

$$\begin{aligned}
\int_{\mathbb{B}} e^{(4\pi-\epsilon)u^{*2}} dx &= \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^{*2}}(1-|x|^2)^2 dv \\
&\ge \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2}(1-|x|^2)^2 dv \\
&= \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx,
\end{aligned}$$

where we have used the Hardy-Littlewood inequality (see [8]) and the fact that the radially decreasing rearrangement of $(1-|x|^2)^2$ with respect to the standard hyperbolic metric is itself. Therefore

$$\sup_{u\in C_0^\infty(\mathbb{B}),\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx \le \sup_{u\in \mathscr{S},\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx.$$

Since $C_0^\infty(\mathbb{B})$ is dense in $\mathscr{H}$, we obtain

$$\sup_{u\in \mathscr{H},\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx = \sup_{u\in \mathscr{S},\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx. \tag{11}$$

To prove (7), we use a method of variation. Observing (10) and (11), we can take a sequence of functions $u_j \in \mathscr{S}$ with $\|u_j\|_{1,\alpha} \le 1$ such that

$$\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_j^2} dx \to \sup_{u\in \mathscr{H},\, \|u\|_{1,\alpha}\le 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx \quad \text{as} \quad j\to\infty. \tag{12}$$

Since $0 \leq \alpha < \lambda_1(\mathbb{B})$, we have
$$\|u_j\|_{\mathscr{H}}^2 \leq \frac{\lambda_1(\mathbb{B})}{\lambda_1(\mathbb{B}) - \alpha}.$$
Note that $\mathscr{H}$ is a Hilbert space. Up to a subsequence there exists some $u_\epsilon \in \mathscr{S}$ such that

$$\begin{array}{lll} u_j \rightharpoonup u_\epsilon & \text{weakly in} & \mathscr{H}, \\ u_j \to u_\epsilon & \text{strongly in} & L^p(\mathbb{B}),\ \forall p \geq 1, \\ u_j \to u_\epsilon & \text{a. e. in} & \mathbb{B}. \end{array}$$

It follows that
$$\|u_\epsilon\|_{1,\alpha}^2 \leq \liminf_{j\to\infty} \|u_j\|_{1,\alpha}^2 \leq 1. \tag{13}$$

A straightforward calculation shows

$$\begin{aligned} \|u_j - u_\epsilon\|_{\mathscr{H}}^2 &= \langle u_j - u_\epsilon, u_j - u_\epsilon \rangle_{\mathscr{H}} \\ &= \|u_j\|_{\mathscr{H}}^2 + \|u_\epsilon\|_{\mathscr{H}}^2 - 2\langle u_j, u_\epsilon \rangle_{\mathscr{H}} \\ &= \|u_j\|_{\mathscr{H}}^2 - \|u_\epsilon\|_{\mathscr{H}}^2 + o_j(1) \\ &= \|u_j\|_{1,\alpha}^2 - \|u_\epsilon\|_{1,\alpha}^2 + o_j(1), \end{aligned} \tag{14}$$

since $\langle u_j, u_\epsilon \rangle_{\mathscr{H}} \to \|u_\epsilon\|_{\mathscr{H}}^2$ and $\|u_j\|_2 \to \|u_\epsilon\|_2$ as $j \to \infty$.

For any $\nu > 0$, using an elementary inequality $2ab \leq \nu a^2 + b^2/\nu$, we have

$$u_j^2 \leq (1+\nu)(u_j - u_\epsilon)^2 + (1 + 1/\nu)u_\epsilon^2.$$

Choosing $\nu = \epsilon/(8\pi - 2\epsilon)$ in the above equation, we have

$$(4\pi - \epsilon)u_j^2 \leq (4\pi - \epsilon/2)(u_j - u_\epsilon)^2 + \frac{32\pi^2}{\epsilon} u_\epsilon^2. \tag{15}$$

From (14) we can find some positive integer j_0 such that

$$\|u_j - u_\epsilon\|_{\mathscr{H}}^2 \leq \frac{4\pi - \epsilon/3}{4\pi - \epsilon/2}, \quad \forall j \geq j_0.$$

This together with (15) gives

$$(4\pi - \epsilon)u_j^2 \leq (4\pi - \epsilon/3)\frac{(u_j - u_\epsilon)^2}{\|u_j - u_\epsilon\|_{\mathscr{H}}^2} + \frac{32\pi^2}{\epsilon} u_\epsilon^2, \quad \forall j \geq j_0. \tag{16}$$

By Lemma 4,
$$\int_{\mathbb{B}} e^{q u_\epsilon^2} dx < \infty, \quad \forall q > 1. \tag{17}$$

Take
$$p = \frac{4\pi - \epsilon/4}{4\pi - \epsilon/3} > 1.$$

Combining (10), (16) and (17), we conclude that $e^{(4\pi-\epsilon)u_j^2}$ is bounded in $L^p(\mathbb{B})$, which together with $u_j \to u_0$ a. e. as $j \to \infty$ implies that $e^{(4\pi-\epsilon)u_j^2}$ converges to $e^{(4\pi-\epsilon)u_\epsilon^2}$ in $L^1(\mathbb{B})$. This together

with (12) leads to (7). Recall (13) we have $\|u_\epsilon\|_{1,\alpha} \leq 1$. Now we claim that $\|u_\epsilon\|_{1,\alpha} = 1$. For otherwise, we have $\|u_\epsilon\|_{1,\alpha} < 1$, and thus

$$\begin{aligned}
\sup_{u\in\mathscr{H},\,\|u\|_{1,\alpha}\leq 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx &= \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx \\
&< \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2/\|u_\epsilon\|_{1,\alpha}^2} dx \\
&\leq \sup_{u\in\mathscr{H},\,\|u\|_{1,\alpha}\leq 1} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx,
\end{aligned}$$

which is a contradiction.

It is not difficult to see that u_ϵ satisfies the Euler-Lagrange equation (8). Finally $u_\epsilon \in C^\infty(\mathbb{B})$ follows from standard elliptic estimates, and the fact that $u_\epsilon \in C^0(\overline{\mathbb{B}})$ follows from $u_\epsilon \in \mathscr{S}$.

Finally we prove (9). Using an elementary inequality $e^t \leq 1 + te^t$ for $t \geq 0$, we have

$$\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx \leq \pi + (4\pi - \epsilon)\lambda_\epsilon.$$

Note that $\int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx$ is monotone with respect to $\epsilon > 0$. For any fixed $u \in \mathscr{H}$ with $\|u\|_{1,\alpha} = 1$, in view of Lemma 4, there holds

$$\pi < \int_{\mathbb{B}} e^{4\pi u^2} dx = \lim_{\epsilon\to 0} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u^2} dx \leq \liminf_{\epsilon\to 0} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx \leq \pi + 4\pi \liminf_{\epsilon\to 0} \lambda_\epsilon.$$

This leads to (9) immediately. □

3.2. Blow-up analysis

In this subsection, we perform the blow-up procedure. Let u_ϵ be as in Proposition 6. Since $\|u_\epsilon\|_{1,\alpha} = 1$ and $\alpha < \lambda_1(\mathbb{B})$, u_ϵ is bounded in $\mathscr{H}$. By Lemma 3, there exists $u_0 \in L^2(\mathbb{B})$ such that up to a subsequence, $u_\epsilon \to u_0$ in $L^2(\mathbb{B})$ and $u_\epsilon \to u_0$ a. e. in $\mathbb{B}$ as $\epsilon \to 0$. On the other hand, there exists some $v_0 \in \mathscr{S}$ such that $u_\epsilon \rightharpoonup v_0$ weakly in $\mathscr{H}$. In particular

$$\lim_{\epsilon\to 0} \int_{\mathbb{B}} u_\epsilon \varphi dx = \int_{\mathbb{B}} v_0 \varphi dx, \quad \forall \varphi \in L^2(\mathbb{B}).$$

Since the weak limit is unique, we have $v_0 = u_0$. In conclusion, there exists $u_0 \in \mathscr{S}$ such that up to a subsequence,

$$u_\epsilon \rightharpoonup u_0 \ \text{ in } \ \mathscr{H}, \ \ u_\epsilon \to u_0 \ \text{ a. e. in } \ \mathbb{B}$$

as $\epsilon \to 0$. Noting that $\langle u_\epsilon, u_0\rangle_{\mathscr{H}} \to \langle u_0, u_0\rangle_{\mathscr{H}}$, we have

$$\|u_0\|_{1,\alpha}^2 = \|u_0\|_{\mathscr{H}}^2 - \alpha\|u_0\|_2^2 \leq \liminf_{\epsilon\to 0} \|u_\epsilon\|_{1,\alpha} = 1.$$

Let $c_\epsilon = u_\epsilon(0) = \max_{\mathbb{B}} u_\epsilon$. If c_ϵ is bounded, we have by using the Lebesgue dominated convergence theorem,

$$\int_{\mathbb{B}} e^{4\pi u_0^2} dx = \lim_{\epsilon\to 0} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx = \sup_{u\in\mathscr{H},\,\|u\|_{1,\alpha}\leq 1} \int_{\mathbb{B}} e^{4\pi u^2} dx.$$

Hence u_0 is a desired extremal function and Theorem 1 holds. In the following, we assume

$$c_\epsilon \to +\infty \text{ as } \epsilon \to 0. \tag{18}$$

Now we claim that $u_0 \equiv 0$. To see this, suppose $u_0 \not\equiv 0$, then $\|u_0\|_{1,\alpha} > 0$. On one hand, by the Hölder inequality, $\forall \nu > 0$, there holds

$$\|e^{(4\pi-\epsilon)u_\epsilon^2}\|_{1+\nu} \leq \|e^{(4\pi-\epsilon)(u_\epsilon-u_0)^2}\|_{(1+\nu)(1+2\nu)}^{1+\nu}\|e^{(4\pi-\epsilon)u_0^2}\|_{(1+\nu)^2(1+2\nu)/\nu^2}^{1+1/\nu}. \tag{19}$$

On the other hand, we calculate

$$\begin{aligned}
\|u_\epsilon - u_0\|_{\mathscr{H}}^2 &= \langle u_\epsilon - u_0, u_\epsilon - u_0\rangle_{\mathscr{H}} \\
&= \|u_\epsilon\|_{\mathscr{H}}^2 - \|u_0\|_{\mathscr{H}}^2 + o_\epsilon(1) \\
&= \|u_\epsilon\|_{1,\alpha}^2 - \|u_0\|_{1,\alpha}^2 + o_\epsilon(1) \\
&= 1 - \|u_0\|_{1,\alpha}^2 + o_\epsilon(1) \\
&< 1 - \|u_0\|_{1,\alpha}^2/2,
\end{aligned} \tag{20}$$

provided that ϵ is sufficiently small. Choosing $\nu = \|u_0\|_{1,\alpha}^2/16$ in (19), we have by (20) and (10) that $e^{(4\pi-\epsilon)u_\epsilon^2}$ is bounded in $L^{1+\nu}(\mathbb{B})$. Then applying standard elliptic estimates to (8), we get that u_ϵ is bounded in $C^0_{\mathrm{loc}}(\mathbb{B})$, which contradicts (18). Therefore $u_0 \equiv 0$.

We set

$$r_\epsilon = \sqrt{\lambda_\epsilon} c_\epsilon^{-1} e^{-(2\pi-\epsilon/2)c_\epsilon^2}.$$

For any $0 < \delta < 4\pi$, we have by using the Hölder inequality and (10),

$$\lambda_\epsilon = \int_{\mathbb{B}} u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2} dx \leq e^{\delta c_\epsilon^2} \int_{\mathbb{B}} u_\epsilon^2 e^{(4\pi-\epsilon-\delta)u_\epsilon^2} dx \leq C e^{\delta c_\epsilon^2}$$

for some constant C depending only on δ. This leads to

$$r_\epsilon^2 \leq C c_\epsilon^{-2} e^{-(4\pi-\epsilon-\delta)c_\epsilon^2} \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{21}$$

Define two blow-up sequences of functions on $\mathbb{B}_{r_\epsilon^{-1}} = \{x \in \mathbb{R}^2 : |x| < r_\epsilon^{-1}\}$ as

$$\psi_\epsilon(x) = c_\epsilon^{-1} u_\epsilon(r_\epsilon x), \quad \varphi_\epsilon(x) = c_\epsilon(u_\epsilon(r_\epsilon x) - c_\epsilon).$$

This kind of blow-up functions are suitable for such a problem was first discovered by Adimurthi-Struwe [5]. A direct computation shows

$$-\Delta\psi_\epsilon = \frac{r_\epsilon^2}{(1-r_\epsilon^2|x|^2)^2}\psi_\epsilon + \alpha r_\epsilon^2 \psi_\epsilon + c_\epsilon^{-2}\psi_\epsilon e^{(4\pi-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon} \quad \text{in} \quad \mathbb{B}_{r_\epsilon^{-1}}, \tag{22}$$

$$-\Delta\varphi_\epsilon = \frac{r_\epsilon^2 c_\epsilon^2}{(1-r_\epsilon^2|x|^2)^2}\psi_\epsilon + \alpha r_\epsilon^2 c_\epsilon^2 \psi_\epsilon + \psi_\epsilon e^{(4\pi-\epsilon)(1+\psi_\epsilon)\varphi_\epsilon} \quad \text{in} \quad \mathbb{B}_{r_\epsilon^{-1}}. \tag{23}$$

We now consider the asymptotic behavior of ψ_ϵ and φ_ϵ. By (21), we have $r_\epsilon^2 c_\epsilon^q \to 0$ as $\epsilon \to 0$ for any $q \geq 1$. Since $\mathbb{B}_{r_\epsilon^{-1}} \to \mathbb{R}^2$ as $\epsilon \to 0$, we have that $|\psi_\epsilon| \leq 1$ and $\Delta\psi_\epsilon(x) \to 0$ uniformly in $\mathbb{B}_R$ for any fixed $R > 0$ as $\epsilon \to 0$. Applying elliptic estimates to (22), we have $\psi_\epsilon \to \psi$ in

$C^1_{\mathrm{loc}}(\mathbb{R}^2)$, where ψ is a bounded harmonic function in $\mathbb{R}^2$. Note that $\psi(0) = \lim_{\epsilon\to 0}\psi_\epsilon(0) = 1$. The Liouville theorem implies that $\psi \equiv 1$ on $\mathbb{R}^2$. Hence

$$\psi_\epsilon \to 1 \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^2).$$

Since $\varphi_\epsilon(x) \le \varphi_\epsilon(0) = 0$ for all $x \in \mathbb{B}_{r_\epsilon^{-1}}$, it is not difficult to see that $\Delta\varphi_\epsilon$ is uniformly bounded in $\mathbb{B}_R$ for any fixed $R > 0$. We then conclude by applying elliptic estimates to the equation (23) that

$$\varphi_\epsilon \to \varphi \quad \text{in} \quad C^1_{\mathrm{loc}}(\mathbb{R}^2), \tag{24}$$

where φ satisfies

$$\begin{cases} \Delta\varphi = -e^{8\pi\varphi} \quad \text{in} \quad \mathbb{R}^2 \\ \varphi(0) = 0 = \sup_{\mathbb{R}^2}\varphi \\ \int_{\mathbb{R}^2} e^{8\pi\varphi}dx \le 1. \end{cases}$$

By a result of Chen-Li [9], we have

$$\varphi(x) = -\frac{1}{4\pi}\log(1+\pi|x|^2), \quad \int_{\mathbb{R}^2} e^{8\pi\varphi}dx = 1. \tag{25}$$

Now we consider the convergence behavior of u_ϵ away from zero. Set $u_{\epsilon,\beta} = \min\{u_\epsilon, \beta c_\epsilon\}$ for any β, $0 < \beta < 1$. Then we have

Lemma 7. *For any β, $0 < \beta < 1$, there holds $\lim_{\epsilon\to 0}\|u_{\epsilon,\beta}\|^2_{1,\alpha} = \beta$.*

Proof. Note that $(u_\epsilon - \beta c_\epsilon)^+ \in W^{1,2}_0(\mathbb{B})$ and thus $u_{\epsilon,\beta} = u_\epsilon - (u_\epsilon - \beta c_\epsilon)^+ \in \mathscr{H}$. Testing the equation (8) by $u_{\epsilon,\beta}$, we have

$$\int_{\mathbb{B}}\left(\nabla u_{\epsilon,\beta}\nabla u_\epsilon - \frac{u_{\epsilon,\beta}u_\epsilon}{(1-|x|^2)^2} - \alpha u_{\epsilon,\beta}u_\epsilon\right)dx = \int_{\mathbb{B}}\frac{1}{\lambda_\epsilon}u_{\epsilon,\beta}u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2}dx.$$

It follows that

$$\begin{aligned} \|u_{\epsilon,\beta}\|^2_{1,\alpha} &= \int_{\mathbb{B}}\left(\nabla u_{\epsilon,\beta}\nabla u_\epsilon - \frac{u_{\epsilon,\beta}u_\epsilon}{(1-|x|^2)^2} - \alpha u_{\epsilon,\beta}u_\epsilon\right)dx \\ &\quad + \int_{\mathbb{B}}\frac{u_{\epsilon,\beta}(u_\epsilon - u_{\epsilon,\beta})}{(1-|x|^2)^2}dx + \alpha\int_{\mathbb{B}}u_{\epsilon,\beta}(u_\epsilon - u_{\epsilon,\beta})dx \\ &\ge \int_{\mathbb{B}}\frac{1}{\lambda_\epsilon}u_{\epsilon,\beta}u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2}dx \\ &\ge \beta\int_{\mathbb{B}_R(0)}(1+o_\epsilon(1))e^{8\pi\varphi}dy \end{aligned}$$

for any $R > 0$. Letting $\epsilon \to 0$ first, and then $R \to \infty$, we obtain

$$\liminf_{\epsilon\to 0}\|u_{\epsilon,\beta}\|^2_{1,\alpha} \ge \beta.$$

Similarly, testing the equation (8) by $(u_\epsilon - \beta c_\epsilon)^+$, we have

$$\begin{aligned}
\|(u_\epsilon - \beta c_\epsilon)^+\|_{1,\alpha}^2 &= \int_{\mathbb{B}} \left(\nabla (u_\epsilon - \beta c_\epsilon)^+ \nabla u_\epsilon - \frac{(u_\epsilon - \beta c_\epsilon)^+ u_\epsilon}{(1-|x|^2)^2} - \alpha (u_\epsilon - \beta c_\epsilon)^+ u_\epsilon \right) dx \\
&\quad + \int_{\mathbb{B}} \frac{(u_\epsilon - \beta c_\epsilon)^+ u_{\epsilon,\beta}}{(1-|x|^2)^2} dx + \alpha \int_{\mathbb{B}} (u_\epsilon - \beta c_\epsilon)^+ u_{\epsilon,\beta} dx \\
&\geq \int_{\mathbb{B}} \frac{1}{\lambda_\epsilon} (u_\epsilon - \beta c_\epsilon)^+ u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2} dx \\
&\geq (1-\beta) \int_{\mathbb{B}_R(0)} (1 + o_\epsilon(1)) e^{8\pi\varphi} dy.
\end{aligned}$$

This implies that

$$\liminf_{\epsilon \to 0} \|(u_\epsilon - \beta c_\epsilon)^+\|_{1,\alpha}^2 \geq (1-\beta).$$

Since $u_\epsilon \to 0$ in $L^p(\mathbb{B})$ as $\epsilon \to 0$ for any fixed $p > 1$, one can see that

$$\lim_{\epsilon \to 0} \left(\|u_{\epsilon,\beta}\|_{1,\alpha}^2 + \|(u_\epsilon - \beta c_\epsilon)^+\|_{1,\alpha}^2 - \|u_\epsilon\|_{1,\alpha}^2 \right) = 0.$$

Therefore

$$\lim_{\epsilon \to 0} \|u_{\epsilon,\beta}\|_{1,\alpha}^2 = \beta, \quad \lim_{\epsilon \to 0} \|(u_\epsilon - \beta c_\epsilon)^+\|_{1,\alpha}^2 = 1 - \beta.$$

This completes the proof of the lemma. □

Lemma 8. *There holds*

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx = \pi + \limsup_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}.$$

Proof. On one hand we have for any β, $0 < \beta < 1$,

$$\begin{aligned}
\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx &= \int_{u_\epsilon < \beta c_\epsilon} e^{(4\pi-\epsilon)u_\epsilon^2} dx + \int_{u_\epsilon \geq \beta c_\epsilon} e^{(4\pi-\epsilon)u_\epsilon^2} dx \\
&\leq \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2} dx + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2}.
\end{aligned}$$

It follows from Lemma 7 that $\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2} dx \to |\mathbb{B}| = \pi$ as $\epsilon \to 0$. Hence

$$\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx \leq \pi + \frac{\lambda_\epsilon}{\beta^2 c_\epsilon^2} + o_\epsilon(1).$$

Letting $\epsilon \to 0$ first, then $\beta \to 1$ in the above inequality, we get

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx \leq \pi + \limsup_{\epsilon \to 0} \frac{\lambda_\epsilon}{c_\epsilon^2}. \tag{26}$$

On the other hand we have by (24)

$$\int_{\mathbb{B}_{Rr_\epsilon}} e^{(4\pi-\epsilon)u_\epsilon^2} dx = \frac{\lambda_\epsilon}{c_\epsilon^2} \left(\int_{\mathbb{B}_R} e^{8\pi\varphi} dx + o_\epsilon(1) \right).$$

It is easy to see that

$$\int_{\mathbb{B}_{Rr_\epsilon}} e^{(4\pi-\epsilon)u_\epsilon^2}dx \leq \int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2}dx - \pi(1-R^2r_\epsilon^2).$$

Combining the above two estimates and letting $\epsilon \to 0$ first, then $R \to +\infty$, we have

$$\limsup_{\epsilon\to 0} \frac{\lambda_\epsilon}{c_\epsilon^2} \leq \lim_{\epsilon\to 0}\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2}dx - \pi. \tag{27}$$

Combining (26) and (27), we get the desired result. □

Obviously Lemma 8 implies that

$$\lim_{\epsilon\to 0} c_\epsilon/\lambda_\epsilon = 0. \tag{28}$$

(Here and in the sequel we do *not* distinguish sequence and subsequence.) This will be used to prove the following:

Lemma 9. *$\forall \phi \in C^\infty(\overline{\mathbb{B}})$, we have*

$$\lim_{\epsilon\to 0}\int_{\mathbb{B}} \phi \frac{1}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2}dx = \phi(0).$$

Proof. For any fixed β, $0<\beta<1$, we divide $\mathbb{B}$ into three parts

$$\mathbb{B} = (\{u_\epsilon > \beta c_\epsilon\} \setminus \mathbb{B}_{Rr_\epsilon}) \cup (\{u_\epsilon \leq \beta c_\epsilon\} \setminus \mathbb{B}_{Rr_\epsilon}) \cup \mathbb{B}_{Rr_\epsilon}.$$

Denote the integrals on the above three domains by I_1, I_2 and I_3 respectively. Firstly we have

$$\begin{aligned} |I_1| &\leq \sup_{\mathbb{B}}|\phi| \int_{\{u_\epsilon>\beta c_\epsilon\}\setminus\mathbb{B}_{Rr_\epsilon}} \frac{1}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2}dx \\ &\leq \frac{1}{\beta}\sup_{\mathbb{B}}|\phi|\left(1-\int_{\mathbb{B}_{Rr_\epsilon}} \frac{1}{\lambda_\epsilon} u_\epsilon^2 e^{(4\pi-\epsilon)u_\epsilon^2}dx\right) \\ &= \frac{1}{\beta}\sup_{\mathbb{B}}|\phi|\left(1-\int_{\mathbb{B}_R} e^{8\pi\varphi}dx + o_\epsilon(R)\right), \end{aligned}$$

where $o_\epsilon(R) \to 0$ as $\epsilon \to 0$ for any fixed $R>0$. Letting $\epsilon \to 0$ first, then $R \to +\infty$, we have $I_1 \to 0$. Secondly there holds

$$|I_2| \leq \sup_{\mathbb{B}}|\phi|\frac{c_\epsilon}{\lambda_\epsilon}\int_{\mathbb{B}} u_\epsilon e^{(4\pi-\epsilon)u_{\epsilon,\beta}^2}dx.$$

It follows from Lemma 7 and (28) that $I_2 \to 0$ as $\epsilon \to 0$ first and then $R \to +\infty$.

Finally we can easily see that

$$I_3 = \phi(\xi)\left(\int_{\mathbb{B}_R} e^{8\pi\varphi}dx + o_\epsilon(R)\right)$$

for some $\xi \in \mathbb{B}_{Rr_\epsilon}$. Letting $\epsilon \to 0$ first, then $R \to +\infty$, we have $I_3 \to \phi(0)$. Combining all the above estimates, we finish the proof of the lemma. □

For simplicity we denote

$$\mathscr{L}_\alpha = -\Delta - \frac{1}{(1-|x|^2)^2} - \alpha.$$

Then we have the following:

Lemma 10. *The function sequence $c_\epsilon u_\epsilon$ converges to G weakly in $W^{1,p}_{\mathrm{loc}}(\mathbb{B})$ for any $p \in (1,2)$, strongly in $L^q(\mathbb{B})$ for any $q \geq 1$, and in $C^0(\overline{\mathbb{B}^c_r})$ for any $r \in (0,1)$, where G is a Green function satisfying $\mathscr{L}_\alpha G = \delta_0$, where δ_0 is the usual Dirac measure centered at* $0 \in \mathbb{B}$.

Proof. Note that

$$\mathscr{L}_\alpha(c_\epsilon u_\epsilon) = f_\epsilon = \frac{1}{\lambda_\epsilon} c_\epsilon u_\epsilon e^{(4\pi-\epsilon)u_\epsilon^2}.$$

Let v_ϵ be a solution to

$$\begin{cases} \mathscr{L}_\alpha v_\epsilon = f_\epsilon \quad \text{in} \quad \mathbb{B}_{1/2} \\ v_\epsilon = 0 \quad \text{on} \quad \partial\mathbb{B}_{1/2} \end{cases} \tag{29}$$

By Lemma 9, f_ϵ is bounded in $L^1(\mathbb{B})$. By a result of Struwe [26], for any q, $1 < q < 2$, there holds

$$\|\nabla v_\epsilon\|_q \leq C\|f_\epsilon\|_1, \tag{30}$$

and there exists some $v_0 \in W^{1,q}_0(\mathbb{B}_{1/2})$ such that

$$v_\epsilon \rightharpoonup v_0 \quad \text{weakly in} \quad W^{1,q}_0(\mathbb{B}_{1/2}). \tag{31}$$

Take a cut-off function $\phi \in C^\infty_0(\mathbb{B})$ satisfying $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $\mathbb{B}_{1/8}$ and $\phi \equiv 0$ outside $\mathbb{B}_{1/4}$. Set $w_\epsilon = c_\epsilon u_\epsilon - \phi v_\epsilon$. It follows that

$$\mathscr{L}_\alpha w_\epsilon = (1-\phi) f_\epsilon + \Delta\phi v_\epsilon + 2\nabla\phi\nabla v_\epsilon.$$

By (28) and Lemma 3, f_ϵ is uniformly bounded in $\mathbb{B} \setminus \mathbb{B}_{1/16}$. While (30) and the Sobolev embedding theorem imply that v_ϵ is bounded in $L^2(\mathbb{B}_{1/2})$. Then applying elliptic estimates to (29), we conclude that v_ϵ is bounded in $W^{2,2}(\mathbb{B}_{1/4} \setminus \mathbb{B}_{1/8})$, and thus $\nabla\phi\nabla v_\epsilon$ is bounded in $L^2(\mathbb{B})$. Therefore $\mathscr{L}_\alpha w_\epsilon$ is bounded in $L^2(\mathbb{B})$. Recalling Lemma 3, we have

$$\|w_\epsilon\|^2_{1,\alpha} = \langle w_\epsilon, \mathscr{L}_\alpha w_\epsilon\rangle_{L^2} \leq C\|w_\epsilon\|_{1,\alpha}\|\mathscr{L}_\alpha w_\epsilon\|_2.$$

This implies that w_ϵ is bounded in $\mathscr{H}$ and there exists some $w_0 \in \mathscr{H}$ such that

$$w_\epsilon \rightharpoonup w_0 \quad \text{weakly in} \quad \mathscr{H}. \tag{32}$$

Let $G = \phi v_0 + w_0$. Here we extend v_0 to be zero in $\mathbb{B} \setminus \mathbb{B}_{1/2}$. It follows from (31) and Lemma 3 that $c_\epsilon u_\epsilon \to G$ in $L^p(\mathbb{B})$ for any $p \geq 1$ and in $C^0(\mathbb{B} \setminus \mathbb{B}_r)$ for any $r > 0$. Moreover we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{B}} c_\epsilon u_\epsilon \mathscr{L}_\alpha\varphi dx = \int_{\mathbb{B}} G\mathscr{L}_\alpha\varphi dx, \quad \forall\varphi \in C^\infty_0(\mathbb{B}).$$

This together with Lemma 9 finishes the proof of the lemma. □

Before ending this subsection, we decompose the Green function G. Since

$$-\Delta\left(G+\frac{1}{2\pi}\log r\right)=\frac{G}{(1-|x|^2)^2}+\alpha G\in L^p_{\rm loc}(\mathbb{B}),$$

there holds

$$G=-\frac{1}{2\pi}\log r+A_0+\widetilde{\psi}, \tag{33}$$

where $\widetilde{\psi}\in C^1_{\rm loc}(\mathbb{B})$.

3.3. *Neck analysis and upper bound estimate*

In this subsection, we use the capacity estimate due to Y. Li [15] to derive an upper bound of the supremum in (4). While in [29], this was done by G. Wang and D. Ye by using a result of Carleson-Chang [10], which was employed originally by Li-Liu-Yang [16] when deriving an upper bound of certain Trudinger-Moser functional for vector bundles on a compact Riemannian surface.

Lemma 11. *For any r, $0<r<1$, there holds*

$$\int_{\mathbb{B}_r}|\nabla u_\epsilon|^2dx=1+\frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log r-A_0+o_r(1)+o_\epsilon(1)\right),$$

where $o_\epsilon(1)\to 0$ as $\epsilon\to 0$, $o_r(1)\to 0$ as $r\to 0$.

Proof. In view of the Euler-Lagrange equation (8), we have by using the divergence theorem

$$\begin{aligned}\int_{\mathbb{B}_r}|\nabla u_\epsilon|^2dx &= -\int_{\mathbb{B}_r}u_\epsilon\Delta u_\epsilon dx+\int_{\partial\mathbb{B}_r}u_\epsilon\frac{\partial u_\epsilon}{\partial\nu}ds\\ &= \int_{\mathbb{B}_r}u_\epsilon\mathscr{L}_\alpha u_\epsilon dx+\int_{\partial\mathbb{B}_r}u_\epsilon\frac{\partial u_\epsilon}{\partial\nu}ds+\int_{\mathbb{B}_r}\frac{u_\epsilon^2}{(1-|x|^2)^2}dx+\alpha\int_{\mathbb{B}_r}u_\epsilon^2dx\\ &= \int_{\mathbb{B}_r}\frac{u_\epsilon^2}{\lambda_\epsilon}e^{(4\pi-\epsilon)u_\epsilon^2}dx+\int_{\partial\mathbb{B}_r}u_\epsilon\frac{\partial u_\epsilon}{\partial\nu}ds+\int_{\mathbb{B}_r}\frac{u_\epsilon^2}{(1-|x|^2)^2}dx+\alpha\int_{\mathbb{B}_r}u_\epsilon^2dx.\end{aligned}$$

Now we estimate the above four integrals respectively. It follows from Lemma 10 and (28) that

$$\int_{\mathbb{B}_r}\frac{u_\epsilon^2}{\lambda_\epsilon}e^{(4\pi-\epsilon)u_\epsilon^2}dx=1-\frac{1}{c_\epsilon^2}\int_{\mathbb{B}\setminus\mathbb{B}_r}\frac{(c_\epsilon u_\epsilon)^2}{\lambda_\epsilon}e^{(4\pi-\epsilon)u_\epsilon^2}dx=1-\frac{1}{c_\epsilon^2}o_\epsilon(1).$$

Moreover, Lemma 10 and (33) lead to

$$\int_{\partial\mathbb{B}_r}u_\epsilon\frac{\partial u_\epsilon}{\partial\nu}ds=\frac{1}{c_\epsilon^2}\left(\int_{\partial\mathbb{B}_r}G\frac{\partial G}{\partial r}ds+o_\epsilon(1)\right)=\frac{1}{c_\epsilon^2}\left(\frac{1}{2\pi}\log r-A_0+o_r(1)\right),$$

$$\int_{\mathbb{B}_r}\frac{u_\epsilon^2}{(1-|x|^2)^2}dx=\frac{1}{c_\epsilon^2}\left(\int_{\mathbb{B}_r}\frac{G^2}{(1-|x|^2)^2}dx+o_\epsilon(1)\right)=\frac{o_r(1)+o_\epsilon(1)}{c_\epsilon^2},$$

and

$$\int_{\mathbb{B}_r}u_\epsilon^2dx=\frac{1}{c_\epsilon^2}\left(\int_{\mathbb{B}_r}G^2dx+o_\epsilon(1)\right)=\frac{o_r(1)+o_\epsilon(1)}{c_\epsilon^2}.$$

Combining all the above estimates, we finish the proof of the lemma. □

Lemma 12. *For two positive numbers δ and R with $\delta > Rr_\epsilon$, there holds*

$$\int_{\mathbb{B}_\delta\setminus\mathbb{B}_{Rr_\epsilon}} |\nabla u_\epsilon|^2 dx = 1 + \frac{1}{c_\epsilon^2}\left(-\frac{\log R}{2\pi} + \frac{\log\delta}{2\pi} - \frac{\log\pi}{4\pi} + \frac{1}{4\pi} - A_0 + o_\delta(1) + O(\frac{1}{R^2}) + o_\epsilon(1)\right).$$

Proof. By (24) and (25), we have

$$\begin{aligned}\int_{\mathbb{B}_{Rr_\epsilon}} |\nabla u_\epsilon|^2 dx &= \frac{1}{c_\epsilon^2}\left(\int_{\mathbb{B}_R} |\nabla\varphi(y)|^2 dy + o_\epsilon(1)\right)\\ &= \frac{1}{c_\epsilon^2}\left(\frac{\log R}{2\pi} + \frac{\log\pi}{4\pi} - \frac{1}{4\pi} + O(\frac{1}{R^2}) + o_\epsilon(1)\right).\end{aligned}$$

This together with Lemma 11 implies the lemma. □

Let $0 < s < r < 1$ and $a, b \in \mathbb{R}$. The function $h : \mathbb{B}_r \setminus \mathbb{B}_s \to \mathbb{R}$ defined by

$$h(x) = \frac{b\log\frac{|x|}{s} + a\log\frac{r}{|x|}}{\log\frac{r}{s}},$$

is harmonic on the planar domain $\mathbb{B}_r \setminus \mathbb{B}_s$. Obviously h has boundary values

$$h|_{\partial\mathbb{B}_s} = a,\ \ h|_{\partial\mathbb{B}_r} = b.$$

Moreover we have

$$\int_{\mathbb{B}_r\setminus\mathbb{B}_s} |\nabla h|^2 dx = \frac{2\pi(b-a)^2}{\log\frac{r}{s}}. \tag{34}$$

Define a function space associated with h as

$$\mathcal{W} = \mathcal{W}(h, r, s) = \left\{u \in W^{1,2}(\mathbb{B}_r \setminus \mathbb{B}_s) \,|\, u - h \in W_0^{1,2}(\mathbb{B}_r \setminus \mathbb{B}_s)\right\}.$$

By a variational direct method, one can see that the infimum

$$\inf_{u\in\mathcal{W}} \int_{\mathbb{B}_r\setminus\mathbb{B}_s} |\nabla u|^2 dx$$

can be attained by the above harmonic function h. In fact we have proved the following:

Lemma 13. *Let $0 < s < r < 1$, $a, b \in \mathbb{R}$, and h, $\mathcal{W}$ be given as above. There holds*

$$\inf_{u\in\mathcal{W}} \int_{\mathbb{B}_r\setminus\mathbb{B}_s} |\nabla u|^2 dx = \frac{2\pi(b-a)^2}{\log\frac{r}{s}}.$$

This lemma can be used to derive the following:

Lemma 14. *Assume $0 < \delta < 1$, $R > 0$ and ϵ is sufficiently small. Then there holds*

$$\int_{\mathbb{B}_\delta\setminus\mathbb{B}_{Rr_\epsilon}} |\nabla u_\epsilon|^2 dx \geq \frac{2\pi(b_\epsilon - a_\epsilon)^2}{\log\frac{\delta}{Rr_\epsilon}},$$

where a_ϵ and b_ϵ are defined as

$$\begin{aligned}a_\epsilon &= u_\epsilon|_{\partial\mathbb{B}_{Rr_\epsilon}} = c_\epsilon + \frac{1}{c_\epsilon}\left(-\frac{1}{2\pi}\log R - \frac{1}{4\pi}\log\pi + O(\frac{1}{R^2}) + o_\epsilon(1)\right),\\ b_\epsilon &= u_\epsilon|_{\partial\mathbb{B}_\delta} = \frac{1}{c_\epsilon}\left(-\frac{1}{2\pi}\log\delta + A_0 + o_\delta(1) + o_\epsilon(1)\right).\end{aligned}$$

Proof. Substitute a_ϵ, b_ϵ, Rr_ϵ and δ for a, b, s and r respectively in Lemma 13. Let

$$h_\epsilon(x) = \frac{b_\epsilon \log \frac{|x|}{Rr_\epsilon} + a_\epsilon \log \frac{\delta}{|x|}}{\log \frac{\delta}{Rr_\epsilon}}.$$

Then $h_\epsilon|_{\partial\mathbb{B}_{Rr_\epsilon}} = u_\epsilon|_{\partial\mathbb{B}_{Rr_\epsilon}}$ and $h_\epsilon|_{\partial\mathbb{B}_\delta} = u_\epsilon|_{\partial\mathbb{B}_\delta}$. Hence we have $u_\epsilon - h_\epsilon \in W_0^{1,2}(\mathbb{B}_\delta \setminus \mathbb{B}_{Rr_\epsilon})$ and

$$\int_{\mathbb{B}_\delta\setminus\mathbb{B}_{Rr_\epsilon}} |\nabla u_\epsilon|^2 dx \geq \inf_{v\in\mathcal{W}(h_\epsilon,\delta,Rr_\epsilon)} \int_{\mathbb{B}_\delta\setminus\mathbb{B}_{Rr_\epsilon}} |\nabla v|^2 dx = \int_{\mathbb{B}_\delta\setminus\mathbb{B}_{Rr_\epsilon}} |\nabla h_\epsilon|^2 dx.$$

This together with an obvious analog of (34) concludes the lemma. □

A straightforward calculation shows

$$\begin{aligned} 2\pi(b_\epsilon - a_\epsilon)^2 &= 2\pi\left\{c_\epsilon + \frac{1}{c_\epsilon}\left(-\frac{1}{2\pi}\log R + \frac{1}{2\pi}\log\delta - \frac{1}{4\pi}\log\pi - A_0 + o(1)\right)\right\}^2 \\ &= 2\pi c_\epsilon^2\left\{1 + \frac{1}{c_\epsilon^2}\left(-\frac{1}{\pi}\log R + \frac{1}{\pi}\log\delta - \frac{1}{2\pi}\log\pi - 2A_0 + o(1)\right)\right\}. \end{aligned} \tag{35}$$

Here and in the sequel $o(1) \to 0$ as $\epsilon \to 0$ first, then $R \to +\infty$, and finally $\delta \to 0$. Also we have

$$\log\frac{\delta}{Rr_\epsilon} = \log\delta - \log R - \log\frac{\sqrt{\lambda_\epsilon}}{c_\epsilon} + (2\pi - \epsilon/2)c_\epsilon^2. \tag{36}$$

Combining Lemma 12, Lemma 14, (35) and (36), we obtain

$$\begin{aligned} &1 + \frac{1}{c_\epsilon^2}\left(-\frac{\log R}{2\pi} + \frac{\log\delta}{2\pi} - \frac{\log\pi}{4\pi} + \frac{1}{4\pi} - A_0 + o(1)\right) \\ &\geq \frac{1 + \frac{1}{c_\epsilon^2}\left(-\frac{\log R}{\pi} + \frac{\log\delta}{\pi} - \frac{\log\pi}{2\pi} - 2A_0 + o(1)\right)}{1 - \frac{\epsilon}{4\pi} + \frac{1}{c_\epsilon^2}\left(-\frac{\log R}{2\pi} + \frac{\log\delta}{2\pi} - \frac{1}{2\pi}\log\frac{\sqrt{\lambda_\epsilon}}{c_\epsilon}\right)}. \end{aligned}$$

This leads to

$$\begin{aligned} &1 + \frac{1}{c_\epsilon^2}\left(-\frac{\log R}{\pi} + \frac{\log\delta}{\pi} - \frac{\log\pi}{4\pi} + \frac{1}{4\pi} - A_0 - \frac{1}{2\pi}\log\frac{\sqrt{\lambda_\epsilon}}{c_\epsilon} + o(1)\right) \\ &\geq 1 + \frac{1}{c_\epsilon^2}\left(-\frac{\log R}{\pi} + \frac{\log\delta}{\pi} - \frac{\log\pi}{2\pi} - 2A_0 + o(1)\right). \end{aligned}$$

It then follows that

$$\frac{1}{2\pi}\log\frac{\sqrt{\lambda_\epsilon}}{c_\epsilon} \leq \frac{\log\pi}{4\pi} + \frac{1}{4\pi} + A_0 + o(1).$$

Therefore

$$\frac{\lambda_\epsilon}{c_\epsilon^2} \leq \pi e^{1+4\pi A_0 + o(1)}.$$

This together with Lemma 8 implies the following:

Proposition 15. *Under the assumption that* $c_\epsilon = \max_{\mathbb{B}} u_\epsilon \to +\infty$ *as* $\epsilon \to 0$, *there holds*

$$\sup_{u\in\mathscr{H},\,\|u\|_{1,\alpha}\leq 1} \int_{\mathbb{B}} e^{4\pi u^2} dx = \lim_{\epsilon\to 0}\int_{\mathbb{B}} e^{(4\pi-\epsilon)u_\epsilon^2} dx \leq \pi + \pi e^{1+4\pi A_0}. \tag{37}$$

3.4. Test function computation

In this subsection, we construct a sequence of test functions $\phi_\epsilon \in \mathscr{H}$ such that $\|\phi_\epsilon\|_{1,\alpha} \leq 1$ and if ϵ is chosen sufficiently small, there holds

$$\int_{\mathbb{B}} e^{4\pi\phi_\epsilon^2} dx > \pi + \pi e^{1+4\pi A_0}. \tag{38}$$

By Proposition 15, this would contradicts (37) unless c_ϵ is bounded. Therefore we get the desired extremal function and complete the proof of Theorem 1.

We set

$$\phi_\epsilon(x) = \begin{cases} c + \frac{-\frac{1}{4\pi}\log(1+\pi\frac{|x|^2}{\epsilon^2})+B}{c} & \text{for} \quad |x| \leq R\epsilon \\ \frac{G(x)}{c} & \text{for} \quad R\epsilon < |x| \leq 1, \end{cases} \tag{39}$$

where $R = -\log\epsilon$, B and c are constants to be determined later. We now require

$$c + \frac{1}{c}\left(-\frac{1}{4\pi}\log(1+\pi R^2) + B\right) = \frac{1}{c}G\,|_{\partial\mathbb{B}_{R\epsilon}} = \frac{1}{c}\left(-\frac{1}{2\pi}\log(R\epsilon) + A_0 + O(R\epsilon)\right), \tag{40}$$

which gives

$$2\pi c^2 = -\log\epsilon - 2\pi B + 2\pi A_0 + \frac{1}{2}\log\pi + O(\frac{1}{R^2}). \tag{41}$$

Clearly, (39) and (40) imply that $\phi_\epsilon \in W^{1,2}_{\mathrm{loc}}(\mathbb{B})$. While in view of (32), G coincides with $w_0 \in \mathscr{H}$ on $\mathbb{B} \setminus \mathbb{B}_{1/2}$. Hence $\phi_\epsilon - w_0/c \in W^{1,2}_0(\mathbb{B})$, which immediately leads to the fact that $\phi_\epsilon \in \mathscr{H}$.

Since $\phi_\epsilon \in \mathscr{H}$, we have by integration by parts, Lemma 10 and (33) that

$$\begin{aligned} \int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} \left(|\nabla\phi_\epsilon|^2 - \frac{\phi_\epsilon^2}{(1-|x|^2)^2} - \alpha\phi_\epsilon^2\right) dx &= \frac{1}{c^2}\int_{\partial\mathbb{B}_{R\epsilon}} G\frac{\partial G}{\partial\nu} ds + \frac{1}{c^2}\int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} G\mathscr{L}_\alpha G dx \\ &= \frac{1}{c^2}\left(-\frac{1}{2\pi}\log(R\epsilon) + A_0 + O(\frac{1}{R^2})\right). \end{aligned}$$

Also a straightforward calculation gives

$$\int_{\mathbb{B}_{R\epsilon}} |\nabla\phi_\epsilon|^2 dx = \frac{1}{c^2}\left(\frac{\log R}{2\pi} + \frac{\log\pi}{4\pi} - \frac{1}{4\pi} + O(\frac{1}{R^2})\right).$$

Hence

$$\begin{aligned} \|\phi_\epsilon\|^2_{1,\alpha} &\leq \int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} \left(|\nabla\phi_\epsilon|^2 - \frac{\phi_\epsilon^2}{(1-|x|^2)^2} - \alpha\phi_\epsilon^2\right) dx + \int_{\mathbb{B}_{R\epsilon}} |\nabla\phi_\epsilon|^2 dx \\ &= \frac{1}{c^2}\left(-\frac{1}{2\pi}\log\epsilon + A_0 + \frac{\log\pi}{4\pi} - \frac{1}{4\pi} + O(\frac{1}{R^2})\right). \end{aligned}$$

We set

$$\int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} \left(|\nabla\phi_\epsilon|^2 - \frac{\phi_\epsilon^2}{(1-|x|^2)^2} - \alpha\phi_\epsilon^2\right) dx + \int_{\mathbb{B}_{R\epsilon}} |\nabla\phi_\epsilon|^2 dx = 1,$$

which implies $\|\phi_\epsilon\|_{1,\alpha} \leq 1$ and

$$2\pi c^2 = -\log\epsilon + 2\pi A_0 + \frac{1}{2}\log\pi - \frac{1}{2} + O(\frac{1}{R^2}). \tag{42}$$

Combining (41) and (42), we obtain

$$B = \frac{1}{4\pi} + O(\frac{1}{R^2}). \tag{43}$$

We now derive the estimate (38). It is clear that

$$\begin{aligned}\int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} e^{4\pi\phi_\epsilon^2}dx &\geq \int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} (1+4\pi\phi_\epsilon^2)dx \\ &= \pi + \frac{4\pi}{c^2}\left(\int_{\mathbb{B}} G^2dx + O(\frac{1}{R^2})\right).\end{aligned}$$

By (42) and (43), there holds on $\mathbb{B}_{R\epsilon}$,

$$\begin{aligned}\phi_\epsilon^2 &\geq c^2 + 2B - \frac{1}{2\pi}\log\left(1+\pi\frac{|x|^2}{\epsilon^2}\right) \\ &= -\frac{1}{2\pi}\log\left(1+\pi\frac{|x|^2}{\epsilon^2}\right) - \frac{1}{2\pi}\log\epsilon + A_0 + \frac{\log\pi}{4\pi} + \frac{1}{4\pi} + O(\frac{1}{R^2}).\end{aligned}$$

This leads to

$$\begin{aligned}\int_{\mathbb{B}_{R\epsilon}} e^{4\pi\phi_\epsilon^2}dx &\geq e^{1+4\pi A_0+\log\pi+O(\frac{1}{R^2})}\int_{\mathbb{B}_R}\frac{1}{(1+\pi|x|^2)^2}dx \\ &= \pi e^{1+4\pi A_0}\left(1+O(\frac{1}{R^2})\right).\end{aligned}$$

Since $1/R^2 = o(1/c^2)$, we obtain

$$\begin{aligned}\int_{\mathbb{B}} e^{4\pi\phi_\epsilon^2}dx &= \int_{\mathbb{B}\setminus\mathbb{B}_{R\epsilon}} e^{4\pi\phi_\epsilon^2}dx + \int_{\mathbb{B}_{R\epsilon}} e^{4\pi\phi_\epsilon^2}dx \\ &\geq \pi + \pi e^{1+4\pi A_0} + \frac{4\pi}{c^2}\left(\int_{\mathbb{B}} G^2dx + o(1)\right).\end{aligned}$$

This gives the desired estimate (38) provided that ϵ is sufficiently small.

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